GLOBAL SOLUTION TO THE THREE-DIMENSIONAL INCOMPRESSIBLE FLOW OF LIQUID CRYSTALS

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Abstract. The equations for the three-dimensional incompressible flow of liquid crystals are considered in a smooth bounded domain. The existence and uniqueness of the global strong solution with small initial data are established. It is also proved that when the strong solution exists, all the global weak solutions constructed in [16] must be equal to the unique strong solution.

1. Introduction

Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid, and those of a solid crystal. For instance, a liquid crystal may flow like a liquid, but its molecules may be oriented in a crystal-like way. There are many different types of liquid crystal phases, which can be distinguished based on their different optical properties. The various liquid crystal phases can be characterized by the type of ordering that is present. One can distinguish positional order and orientational order, and moreover order can be either short-range or long-range. Liquid crystals may have an isotropic phase at high temperature, or anisotropic orientational structure at lower temperature. The diverse phases of liquid crystals have wide applications from the liquid crystal display to biology (In particular, biological membranes and cell membranes are a form of liquid crystal). In the 1960s, the theoretical physicist P.-G. de Gennes found fascinating analogies between liquid crystals and superconductors as well as magnetic materials, which was rewarded with the Nobel Prize in Physics in 1991. One of the most common liquid crystal phases is the nematic, where the molecules have no positional order, but they have long-range orientational order. For more details of physics, we refer the readers to the two books of de Gennes-Prost [5] and Chandrasekhar [3].

The three-dimensional flow of nematic liquid crystals can be governed by the following system of partial differential equations ([5, 14, 15, 16]):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = -\lambda \mathrm{div}\left(\nabla\mathbf{d}\odot\nabla\mathbf{d}\right), \tag{1.1a}$$

$$\frac{\partial \mathbf{d}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{d} = \gamma\left(\Delta\mathbf{d} - f(\mathbf{d})\right), \tag{1.1b}$$

$$\mathrm{div}\mathbf{u} = 0. \tag{1.1c}$$

where $\mathbf{u}\in\mathbb{R}^3$ denotes the vector field, $\mathbf{d}\in\mathbb{R}^3$ the director field for the averaged macroscopic molecular orientations, $P\in\mathbb{R}$ the pressure arising from the incompressibility; and

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they all depend on the spatial variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the time variable $t > 0$. The positive constants μ, λ, γ stand for viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Debroah number for the molecular orientation field, respectively. We set these three constants to be one since their sizes do not play any role in our analysis. The symbol $\nabla\mathbf{d} \odot \nabla\mathbf{d}$ denotes a matrix whose ij-th entry is $< \partial_{x_i}\mathbf{d}, \partial_{x_j}\mathbf{d} >$, and it is easy to see that

$$\nabla\mathbf{d} \odot \nabla\mathbf{d} = (\nabla\mathbf{d})^\top \nabla\mathbf{d},$$

where $(\nabla\mathbf{d})^\top$ denotes the transpose of the 3×3 matric $\nabla\mathbf{d}$. In (1.1), $f(\mathbf{d})$ is the penalty function which will be assumed to be zero as in [16] for the three-dimensional problem. The system (1.1) is a simplified version, but still retains most of the essential features, of the Ericksen-Leslie equations ([7, 8, 10, 11, 12, 13]) for the hydrodynamics of nematic liquid crystals; see [16, 19, 20] for more discussions on the relations of the two models. Both the Ericksen-Leslie system and the simplified one (1.1) describe the time evolution of liquid crystal materials under the influence of both the velocity field $\mathbf{u}$ and the director field $\mathbf{d}$. In many situations, the flow velocity field does disturb the alignment of the molecule, and a change in the alignment will induce velocity.

We consider the initial-boundary value problem of system (1.1) in a bounded domain $\Omega \subset \mathbb{R}^3$ with C^3 boundary under the initial-boundary conditions:

$$\mathbf{d}|_{t=0} = \mathbf{d}_0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{1.2}$$

and

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{d}_0, \tag{1.3}$$

with $\mathrm{div}\mathbf{u}_0 = 0$ in Ω, and $\mathbf{d}_0 \in C^1(\overline{\Omega})$ satisfying $\nabla\mathbf{d}_0 = 0$ on the boundary $\partial\Omega$. We introduce an 3×3 matrix

$$\mathtt{F} = \nabla\mathbf{d}, \tag{1.4}$$

and take the gradient of (1.1b) to rewrite (1.1), with $f(\mathbf{d}) = 0$ and $\mu = \lambda = \gamma = 1$, as:

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\mathrm{div}(\mathtt{F}^\top\mathtt{F}), \tag{1.5a}$$

$$\frac{\partial\mathtt{F}}{\partial t} + \mathbf{u}\cdot\nabla\mathtt{F} + \mathtt{F}\nabla\mathbf{u} = \Delta\mathtt{F}, \tag{1.5b}$$

$$\mathrm{div}\mathbf{u} = 0, \tag{1.5c}$$

where we used, for all $i, j, k = 1, 2, 3$,

$$\frac{\partial}{\partial x_k}\left(\mathbf{u}_j\frac{\partial\mathbf{d}_i}{\partial x_j}\right) = \frac{\partial\mathbf{u}_j}{\partial x_k}\frac{\partial\mathbf{d}_i}{\partial x_j} + \mathbf{u}_j\frac{\partial}{\partial x_j}\left(\frac{\partial\mathbf{d}_i}{\partial x_k}\right) = (\mathtt{F}\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathtt{F})_{ik}.$$

Notice that (1.5a) is the incompressible Navier-Stokes equation with the source term, $-\mathrm{div}(\mathtt{F}^\top\mathtt{F})$, while (1.5b) is a parabolic equation of $\mathtt{F}$. The initial-boundary conditions (1.2) and (1.3) become

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathtt{F}|_{t=0} = \mathtt{F}_0 := \nabla\mathbf{d}_0, \tag{1.6}$$

and

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathtt{F}|_{\partial\Omega} = 0. \tag{1.7}$$

There have been some studies on system (1.1). In Lin-Liu [16], the global existence of weak solutions with large initial data was proved under the condition that the orientational configuration $\mathbf{d}(x, t)$ belongs to H^2, and the global existence of classical solutions was also

obtained if the coefficient μ is large enough in three dimensional spaces. The similar results were obtained also in [20] for a different but similar model. When weak solutions are discussed, the regularity of the weak solution was investigated in [17] (and also [11]).

In this paper, we are interested in strong solutions of (1.5) in the Sobolev space $W^{2,q}(\Omega)$ with $q > 3$. It is worthy of pointing out that if $\mathtt{F}$ belongs to $W^{2,q}(\Omega)$, it is equivalent to saying that $\mathbf{d}$ should be in $W^{3,q}(\Omega)$ according to (1.4). By a *Strong Solution*, we means a triplet $(\mathbf{u}, \mathtt{F}, P)$ satisfying (1.5) almost everywhere with the initial condition (1.6) and the boundary condition (1.7). Our strategy to consider (1.5) in $W^{2,q}(\Omega)$ is to linearize (1.5) as

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla P = -\mathbf{v}\cdot\nabla\mathbf{v} - \mathrm{div}(G^\top G), \tag{1.8a}$$

$$\frac{\partial \mathtt{F}}{\partial t} - \Delta \mathtt{F} = -\mathbf{v}\cdot\nabla G - G\nabla\mathbf{v}, \tag{1.8b}$$

$$\mathrm{div}\mathbf{u} = 0, \tag{1.8c}$$

for some given $\mathbf{v} \in \mathbb{R}^3$ and $G \in M^{3\times 3}$. One of the motivations of making such an linearization is that we can use the maximal regularity of Stokes equations ([4]) and the parabolic equation ([1]). We first use an iteration method to establish the local existence and uniqueness of strong solution with general initial data. Then we prove the global existence by establishing some global estimates under the condition that the initial data is small in some sense. The global weak solution was obtained in Lin-Liu [16], but the uniqueness is still an open problem. We shall prove that when the strong solution exists, all the global weak solutions constructed in [16] must be equal to the unique strong solution, which is called the weak-strong uniqueness. Similar results were obtained by Danchin [4] for the density-dependent incompressible Navier-Stokes equations. We shall establish our results in the spirit of [4], while developing new estimates for the director field $\mathbf{d}$.

The rest of the paper is organized as follows. In Section 2, we state our main results on local and global existence of strong solution, as well as the weak-strong uniqueness. In Section 3, we recall the maximal regularity for Stokes equations and the parabolic equation, and also some L^∞ estimates. In Section 4, we give the proof of the local existence. In Section 5, we prove the global existence. Finally in Section 6, we show the weak-strong uniqueness.

2. Main Results

In this section, we state our main results. If $k > 0$ is an integer and $p \geq 1$, we denote by $W^{k,p}$ the set of functions in $L^p(\Omega)$ whose derivatives of up to order k belong to $L^p(\Omega)$. For $T > 0$ and a function space X, denote by $L^p(0,T;X)$ the set of Bochner measurable X-valued time dependent functions f such that $t \to \|f\|_X$ belongs to $L^p(0,T)$. Let us define the functional spaces in which the existence of solutions is going to be obtained:

Definition 2.1. For $T > 0$ and $1 < p, q < \infty$, we denote by $M_T^{p,q}$ the set of triplets $(\mathbf{u}, \mathtt{F}, P)$ such that

$$\mathbf{u} \in C([0,T]; D_{A_q}^{1-\frac{1}{p},p}) \cap L^p(0,T; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)), \quad \partial_t \mathbf{u} \in L^p(0,T;L^q), \quad \mathrm{div}\mathbf{u} = 0,$$

$$\mathtt{F} \in C([0,T]; B_{q,p}^{2(1-\frac{1}{p})} \cap L^p(0,T; W^{2,q}(\Omega)), \quad \partial_t \mathtt{F} \in L^p(0,T; L^q(\Omega)),$$

and
$$P \in L^p(0,T;W^{1,q}(\Omega)), \quad \int_\Omega P dx = 0.$$
The corresponding norm is denoted by $\|\cdot\|_{M_T^{p,q}}$.

In the above definition, the space $D_{A_q}^{1-\frac{1}{p},p}$ stands for some fractional domain of the Stokes operator in L^q (cf. Section 2.3 in [4]). Roughly, the vector-fields of $D_{A_q}^{1-\frac{1}{p},p}$ are vectors which have $2-\frac{2}{p}$ derivatives in L^q, are divergence-free, and vanish on $\partial\Omega$. The Besov space (for definition, see [2]) $B_{q,p}^{2(1-\frac{1}{p})}$ can be regarded as the interpolation space between L^q and $W^{2,q}$, that is,
$$B_{q,p}^{2(1-\frac{1}{p})} = (L^q, W^{2,q})_{1-\frac{1}{p},p}.$$
We note that, from Proposition 2.5 in [4],
$$D_{A_q}^{1-\frac{1}{p},p} \hookrightarrow B_{q,p}^{2(1-\frac{1}{p})} \cap L^q(\Omega). \tag{2.1}$$

The local existence will be shown by using an iterative method, and if the initial data is sufficiently small in some suitable function spaces, the solution is indeed global in time. More precisely, our existence results read:

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with C^3 boundary. Assume $1 \le p,q \le \infty$ with $\frac{2}{p}(1-\frac{3}{q}) \in (0,1)$ and $\mathbf{u}_0 \in D_{A_q}^{1-\frac{1}{p},p}$, $\mathtt{F}_0 \in B_{q,p}^{2(1-\frac{1}{p})} \cap L^q$. Then,*

(1) *There exists a $T_0 > 0$, such that, system* (1.5) *with the initial-boundary conditions* (1.6)-(1.7) *has a unique local strong solution $(\mathbf{u}, \mathtt{F}, P) \in M_{T_0}^{p,q}$ in $\Omega \times (0,T_0)$;*

(2) *Moreover, there exists a $\delta_0 > 0$, such that, if the initial data satisfies*
$$\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} \le \delta_0, \quad \|\mathtt{F}_0\|_{B_{q,p}^{2(1-\frac{1}{p})} \cap L^q} \le \delta_0,$$
then (1.6)-(1.7) *has a unique global strong solution $(\mathbf{u}, \mathtt{F}, P) \in M_T^{p,q}$ in $\Omega \times (0,T)$ for all $T > 0$.*

Remark 2.1. The above Theorem gives us the global strong solution near $\mathbf{u} = 0$, $\mathtt{F} = 0$. The similar argument to the proof of Theorem 2.1 below will also enable us to show the global existence of strong solution to (1.5) near the equilibrium state: $\mathbf{u} = 0$, $\mathtt{F} = I$ (the 3×3 identity matrix).

According to Lin-Liu [16], for the given initial-boundary conditions (1.6) and (1.7), there exists at least a *Weak Solution* to (1.5). But its uniqueness is still an open question. More precisely, a triplet (v, E, Π) is called a weak solution to (1.5) with (1.6) and (1.7) in $\Omega \times (0,T)$ if (v, E, Π) satisfies the system (1.5) in the sense of distributions, i.e, for all $\psi \in (C_0^\infty(\Omega \times (0,T)))^3$ with $\mathrm{div}\psi = 0$ and $\phi \in (C_0^\infty(\Omega \times (0,T)))^9$, we have
$$\int_0^T\!\!\int_\Omega v\partial_t\psi\, dxdt + \int_0^T\!\!\int_\Omega v \otimes v : \nabla\psi\, dxdt - \int_0^T\!\!\int_\Omega \nabla v : \nabla\psi\, dxdt = -\int_0^T\!\!\int_\Omega E^\top E : \nabla\psi\, dxdt;$$
and
$$\int_0^T E : \partial_t\phi\, dxdt - \int_0^T\!\!\int_\Omega \mathbf{u}\cdot\nabla E : \phi\, dxdt - \int_0^T\!\!\int_\Omega E\nabla\mathbf{u} : \phi\, dxdt = \int_0^T\!\!\int_\Omega \nabla E : \nabla\phi\, dxdt,$$

with the energy inequality:

$$\int_\Omega (|v(t)|^2 + |E(t)|^2)dx + \int_0^t \int_\Omega (|\nabla v|^2 + |\nabla E|^2)dxds \le \int_\Omega (|v_0|^2 + |E_0|^2)dx.$$

In this weak formulation, the pressure Π can be determined as in the Navier-Stokes equations, see Galdi [9]. We state here the existence of weak solutions in Theorem A of [16]:

Proposition 2.1. *Assume that* $\mathbf{u}_0 \in L^2$ *and* $\mathtt{F}_0 \in L^2$. *Then the system* (1.5) *with the initial condition* (1.6) *and the boundary condition* (1.7) *has a global weak solution* (v, E, Π) *such that*

$$v \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2),$$

and

$$E \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2),$$

for all $T \in (0, \infty)$.

For the same initial-boundary conditions, the relation between its weak solution and its strong solution can be formulated as:

Theorem 2.2. *Assume that* $\mathbf{u}_0 \in D_{A_q}^{1-\frac{1}{p},p}$ *and* $\mathtt{F}_0 \in B_{q,p}^{2(1-\frac{1}{p})} \cap L^q$. *Then its corresponding weak solution to* (1.5) *with* (1.6) *and* (1.7) *is unique and indeed is equal to its unique strong solution.*

Usually, we call this kind of uniqueness as *Weak-Strong Uniqueness*. For the similar results on the compressible Navier-Stokes equations, we refer the readers to [6, 18].

3. Maximal Regularity

In this section, we recall the maximal regularities for the parabolic operator and the Stokes operator, as well as some L^∞ estimates. For $T > 0$, $1 < p, q < \infty$, denote

$$\mathcal{W}(0,T) := W^{1,p}(0,T;(L^q(\Omega))^3) \cap L^p(0,T;(W^{2,q}(\Omega))^3).$$

Throughout this paper, C stands for a generic positive constant.

We first recall the maximal regularity for the parabolic operator (cf. Theorem 4.10.7 and Remark 4.10.9 in [1]):

Theorem 3.1. *Given* $1 < p < \infty$, $\omega_0 \in B_{q,p}^{2(1-\frac{1}{p})}$ *and* $f \in L^p(0,T;L^q(\mathbb{R}^3)^3)$, *the Cauchy problem*

$$\frac{d\omega}{dt} - \Delta\omega = f, \quad t \in (0,T), \quad \omega(0) = \omega_0,$$

has a unique solution $\omega \in \mathcal{W}(0,T)$, *and*

$$\|\omega\|_{\mathcal{W}(0,T)} \le C \left(\|f\|_{L^p(0,T;L^q(\mathbb{R}^3))} + \|\omega_0\|_{B_{q,p}^{2(1-\frac{1}{p})}} \right),$$

where C *is independent of* ω_0, f *and* T. *Moreover, there exists a positive constant* c_0 *independent of* f *and* T *such that*

$$\|\omega\|_{\mathcal{W}(0,T)} \ge c_0 \sup_{t\in(0,T)} \|\omega(t)\|_{B_{q,p}^{2(1-\frac{1}{p})}}.$$

Now we recall the maximal regularity for the Stokes equations (cf. Theorem 3.2 in [4]):

Theorem 3.2. *Let Ω be a bounded domain with C^3 boundary in $\mathbb{R}^3$ and $1 < p, q < \infty$. Assume that $\mathbf{u}_0 \in D_{A_q}^{1-\frac{1}{p},p}$ and $f \in L^p(\mathbb{R}^+; L^q)$. Then the system*

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla P = f, \quad \int_\Omega P dx = 0, \\ \mathrm{div}\mathbf{u} = 0, \quad \mathbf{u}|_{\partial\Omega} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases}$$

has a unique solution $(\mathbf{u}, P)$ satisfying the following inequality for all $T > 0$:

$$\begin{aligned} &\|\mathbf{u}(T)\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \|(\nabla P, \Delta \mathbf{u}, \partial_t \mathbf{u})\|_{L^q}^p \, dt \right)^{\frac{1}{p}} \\ &\qquad \le C \left(\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \|f(t)\|_{L^q}^p dt \right)^{\frac{1}{p}} \right) \end{aligned} \tag{3.1}$$

with $C = C(q, p, \Omega)$.

Remark 3.1. We notice that (3.1) does not include the estimate for $\|\mathbf{u}\|_{L^p(0,T;L^q)}$. Indeed, thanks to $\mathbf{u}|_{\partial\Omega} = 0$, Poincare's inequality, and the fact $\int_\Omega \nabla \mathbf{u} dx = 0$, we have

$$\|\mathbf{u}\|_{W^{2,q}} \le C \|\Delta \mathbf{u}\|_{L^q},$$

and then (3.1) can be rewritten as

$$\begin{aligned} &\|\mathbf{u}(T)\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \|(\nabla P, \mathbf{u}, \Delta \mathbf{u}, \partial_t \mathbf{u})\|_{L^q}^p \, dt \right)^{\frac{1}{p}} \\ &\qquad \le C \left(\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \|f(t)\|_{L^q}^p dt \right)^{\frac{1}{p}} \right). \end{aligned} \tag{3.2}$$

We have the L^∞ estimate in the spatial variable as follows (cf. Lemma 4.1 in [4]).

Lemma 3.1. *Let $1 < p, q, r, s < \infty$ satisfy*

$$0 < \frac{p}{2} - \frac{3p}{2r} < 1, \quad \frac{1}{s} = \frac{1}{r} + \frac{1}{q}.$$

Then the following inequalities hold:

$$\|\nabla f\|_{L^p(0,T;L^\infty)} \le C T^{\frac{1}{2} - \frac{3}{2r}} \|f\|_{L^\infty(0,T;D_{A_r}^{1-\frac{1}{p},p})}^{1-\theta} \|f\|_{L^p(0,T;W^{2,r})}^\theta,$$

$$\|\nabla f\|_{L^p(0,T;L^q)} \le C T^{\frac{1}{2} - \frac{3}{2r}} \|f\|_{L^\infty(0,T;D_{A_s}^{1-\frac{1}{p},p})}^{1-\theta} \|f\|_{L^p(0,T;W^{2,s})}^\theta,$$

for some constant C depending only on Ω, p, q and

$$\frac{1-\theta}{p} = \frac{1}{2} - \frac{3}{2r}.$$

Similarly, we have,

Lemma 3.2. *Let $1<p,q<\infty$ satisfy $0<\frac{p}{2}-\frac{3p}{2q}<1$. Then one has,*

$$\|\nabla f\|_{L^p(0,T;L^\infty)}\le CT^{\frac12-\frac{3}{2q}}\|f\|^{1-\theta}_{L^\infty(0,T;B^{2(1-\frac1p),p}_{q,p})}\|f\|^{\theta}_{L^p(0,T;W^{2,q})},$$

for some constant C depending only on Ω,p,q and

$$\frac{1-\theta}{p}=\frac12-\frac{3}{2q}.$$

Proof. First, we notice that

$$(B^{1-\frac2p-\frac3q}_{\infty,\infty},B^{1-\frac3q}_{\infty,\infty})_{\theta,1}=B^0_{\infty,1}$$

with

$$\frac{1-\theta}{p}=\frac12-\frac{3}{2q},$$

see Theorem 6.4.5 in [2]. Also the imbedding $B^0_{\infty,1}\hookrightarrow L^\infty$ is true due to Theorem 6.2.4 in [2]. Hence, one has

$$\|\nabla f\|_{L^\infty}\le C\|\nabla f\|_{B^0_{\infty,1}}\le C\|\nabla f\|^{\theta}_{B^{1-\frac3q}_{\infty,\infty}}\|\nabla f\|^{1-\theta}_{B^{1-\frac2p-\frac3q}_{\infty,\infty}}. \tag{3.3}$$

We remark that

$$B^{2(1-\frac1p)}_{q,p}\hookrightarrow B^{2-\frac2p-\frac3q}_{\infty,\infty}\hookrightarrow B^{1-\frac2p-\frac3q}_{\infty,\infty},\quad W^{1,q}\hookrightarrow B^{1-\frac3q}_{\infty,\infty},$$

see Theorem 6.2.4 and Theorem 6.5.1 in [2]. Hence, according to (3.3), one deduce that

$$\begin{aligned}\|\nabla f\|_{L^p(0,T;L^\infty)}&\le C\left(\int_0^T\|\nabla f\|^{p\theta}_{B^{1-\frac3q}_{\infty,\infty}}\|\nabla f\|^{p(1-\theta)}_{B^{1-\frac2p-\frac3q}_{\infty,\infty}}dt\right)^{\frac1p}\\&\le C\left(\int_0^T\|f\|^{p\theta}_{W^{2,q}}\|f\|^{p(1-\theta)}_{B^{2(1-\frac1p),p}_{q,p}}dt\right)^{\frac1p}\\&\le CT^{\frac12-\frac{3}{2q}}\|f\|^{1-\theta}_{L^\infty(0,T;B^{2(1-\frac1p),p}_{q,p})}\|f\|^{\theta}_{L^p(0,T;W^{2,q})}.\end{aligned}$$

□

Lemma 3.3. *For $f\in L^p(0,T,L^q)$ and $\partial_tf\in L^p(0,T;L^q)$ with $f(0)\in L^q$, we have, for all $t\in[0,T]$,*

$$\|f\|_{L^\infty(0,t;L^q)}\le C\left(\|f_0\|_{L^q}+\|f\|_{L^p(0,t;L^q)}+\|\partial_tf\|_{L^p(0,t;L^q)}\right), \tag{3.4}$$

for some positive constant C independent of T and f.

Proof. Indeed, we have

$$\begin{aligned}
\|f(t)\|_{L^q}^p &= \|f_0\|_{L^q}^p + \int_0^t \frac{d}{ds}\|f(s)\|_{L^q}^p ds \\
&= \|f_0\|_{L^q}^p + \frac{p}{q}\int_0^t \left(\|f(t)\|_{L^q}^{p-q} \int_{\mathbb{R}^3} |f(s)|^{q-2} f(s)\partial_s f(t) dx \right) dt \\
&\leq \|f_0\|_{L^q}^p + \frac{p}{q}\int_0^t \|f(s)\|_{L^q}^{p-1} \|\partial_s f\|_{L^q} ds \\
&\leq \|f_0\|_{L^q}^p + \frac{p}{q}\left(\int_0^t \|f\|_{L^q}^p ds\right)^{\frac{p-1}{p}} \left(\int_0^t \|\partial_s f\|_{L^q}^p ds\right)^{\frac{1}{p}},
\end{aligned}$$

and consequently, (3.4) follows from Hölder's inequality. □

4. Local Existence

In this section, we prove the local existence and uniqueness of strong solution in Theorem 2.1. The proof will be divided into several steps, including constructing the approximate solution by iteration, obtaining the uniform estimate, showing the convergence, consistency, and uniqueness.

4.1. **Construction of approximate solutions.** We initialize the construction of approximate solutions by setting $F^0 := \mathtt{F}_0$ and $\mathbf{u}^0 := \mathbf{u}_0$. For given $(\mathbf{u}^n, \mathtt{F}^n)$, the Stokes equations (1.8a) and the parabolic equation (1.8b) enable us to define $(\mathbf{u}^{n+1}, \mathtt{F}^{n+1}, P^{n+1})$ as the global solution of

$$\frac{\partial \mathbf{u}^{n+1}}{\partial t} - \Delta \mathbf{u}^{n+1} + \nabla P^{n+1} = -\mathbf{u}^n \cdot \nabla \mathbf{u}^n - \mathrm{div}(\mathtt{F}^{n\top}\mathtt{F}^n), \tag{4.1a}$$

$$\frac{\partial \mathtt{F}^{n+1}}{\partial t} - \Delta \mathtt{F}^{n+1} = -\mathbf{u}^n \cdot \nabla \mathtt{F}^n - \mathtt{F}^n \nabla \mathbf{u}^n, \tag{4.1b}$$

$$\mathrm{div}\mathbf{u}^{n+1} = 0, \tag{4.1c}$$

with the initial-boundary conditions:

$$\mathbf{u}^{n+1}|_{t=0} = \mathbf{u}_0, \quad \mathtt{F}^{n+1}|_{t=0} = \mathtt{F}_0,$$

$$\mathbf{u}^{n+1}|_{\partial\Omega} = 0, \quad \mathtt{F}^{n+1}|_{\partial\Omega} = 0,$$

and

$$\int_\Omega P^{n+1} dx = 0.$$

According to Theorem 3.1 and Theorem 3.2, an argument by induction yields a sequence $\{(\mathbf{u}^n, \mathtt{F}^n, P^n)\}_{n\in\mathbb{N}} \subset M_T^{p,q}$ for all positive T.

4.2. **Uniform estimate for some small fixed time T.** We aim at finding a positive time T independent of n for which $\{(\mathbf{u}^n, \mathtt{F}^n, P^n)\}_{n\in\mathbb{N}}$ is uniformly bounded in the space $M_T^{p,q}$. Indeed, applying Theorem 3.1 and Theorem 3.2, we obtain

$$\begin{aligned}
&\|\mathbf{u}^{n+1}(T)\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \left\|(\nabla P^{n+1}, \mathbf{u}^{n+1}, \Delta\mathbf{u}^{n+1}, \partial_t\mathbf{u}^{n+1})\right\|_{L^q}^p dt\right)^{\frac{1}{p}} \\
&\leq C\left(\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \|\mathbf{u}^n\cdot\nabla\mathbf{u}^n + \operatorname{div}(\mathtt{F}^{n\top}\mathtt{F}^n)\|_{L^q}^p dt\right)^{\frac{1}{p}}\right),
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\|\mathtt{F}^{n+1}(T)\|_{B_{q,p}^{2(1-\frac{1}{p})}} + \|\mathtt{F}^{n+1}\|_{\mathcal{W}(0,T)} \\
&\leq C\left(\|F_0\|_{B_{q,p}^{2(1-\frac{1}{p})}} + \|\mathtt{F}^n\nabla\mathbf{u}^n + \mathbf{u}^n\cdot\nabla\mathtt{F}^n\|_{L^p(0,T;L^q(\Omega))}\right).
\end{aligned} \tag{4.3}$$

Now define

$$\begin{aligned}
U^n(t) :=& \|\mathbf{u}^n(t)\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} + \|\mathbf{u}^n\|_{L^p(0,t;W^{2,q})} + \|\partial_t\mathbf{u}^n\|_{L^p(0,t;L^q)} \\
&+ \|\mathtt{F}^n(t)\|_{L^\infty(0,t;B_{q,p}^{2(1-\frac{1}{p})})} + \|\mathtt{F}^n\|_{\mathcal{W}(0,t)},
\end{aligned}$$

and

$$U^0 = \|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \|F_0\|_{B_{q,p}^{2(1-\frac{1}{p})}\cap L^q}.$$

Hence, from (4.2) and (4.3), one has, using Lemmas 3.1-3.3,

$$\begin{aligned}
U^{n+1}(t) \leq& C\Big(U^0 + \|\mathtt{F}^n\|_{L^\infty(0,t;L^q)}\|\nabla\mathtt{F}^n\|_{L^p(0,t;L^\infty)} + \|\mathbf{u}^n\|_{L^\infty(0,t;L^q)}\|\nabla\mathbf{u}^n\|_{L^p(0,t;L^\infty)} \\
&+ \|\mathbf{u}^n\|_{L^\infty(0,t;L^q)}\|\nabla\mathtt{F}^n\|_{L^p(0,t;L^\infty)} + \|\mathtt{F}^n\|_{L^\infty(0,t;L^q)}\|\nabla\mathbf{u}^n\|_{L^p(0,T;L^\infty)}\Big) \\
\leq& C\left(U^0 + t^{\frac{1}{2}-\frac{3}{2q}}(U^0 + U^n(t))U^n(t)\right).
\end{aligned} \tag{4.4}$$

Hence, if we assume that $U^n(t) \leq 4CU^0$ on $[0, T_0]$ with

$$0 < T_0 \leq \left(\frac{3}{4C(4C+1)U^0}\right)^{\frac{2q}{q-3}}, \tag{4.5}$$

then a direct computation yields

$$U^{n+1}(t) \leq 4CU^0, \quad \text{on} \quad [0, T_0].$$

Coming back to (4.2), (4.3), and (4.4), we conclude that the sequence $\{(\mathbf{u}^n, \mathtt{F}^n, P^n)\}_{n=1}^\infty$ is uniformly bounded in $M_{T_0}^{p,q}$. More precisely, we have

Lemma 4.1. *For all $t \in [0, T_0]$ with T_0 satisfying (4.5),*

$$U^n(t) \leq 4CU^0. \tag{4.6}$$

4.3. **Convergence of the approximate sequence.** We now prove

Lemma 4.2. *$\{(\mathbf{u}^n, \mathtt{F}^n, P^n)\}_{n=1}^\infty$ is a Cauchy sequence and thus converges in $M_{T_0}^{p,q}$.*

Proof. Let

$$\delta\mathbf{u}^n := \mathbf{u}^{n+1} - \mathbf{u}^n, \quad \delta P^n := P^{n+1} - P^n, \quad \delta\mathtt{F}^n := \mathtt{F}^{n+1} - \mathtt{F}^n.$$

Define

$$\begin{aligned} \delta U^n(t) :=& \|\delta\mathbf{u}^n(t)\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} + \|\delta\mathbf{u}^n\|_{L^p(0,t;W^{2,q})} + \|\partial_t\delta\mathbf{u}^n\|_{L^p(0,t;L^q)} \\ &+ \|\delta\mathtt{F}^n(t)\|_{L^\infty(0,t;B_{q,p}^{2(1-\frac{1}{p})})} + \|\delta\mathtt{F}^n\|_{\mathcal{W}(0,t)}. \end{aligned} \tag{4.7}$$

The triplet $(\delta\mathbf{u}^n, \delta\mathtt{F}^n, \delta P^n)$ satisfies

$$\begin{cases} \frac{\partial\delta\mathbf{u}^n}{\partial t} - \Delta\delta\mathbf{u}^n + \nabla\delta P^n \\ \qquad\qquad = -\mathbf{u}^n\cdot\nabla\mathbf{u}^n + \mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^{n-1} - \mathrm{div}(\mathtt{F}^{n\top}\mathtt{F}^n) + \mathrm{div}(\mathtt{F}^{n-1\top}\mathtt{F}^{n-1}), \\ \frac{\partial\delta\mathtt{F}^n}{\partial t} - \Delta\delta\mathtt{F}^n = -\mathbf{u}^n\cdot\nabla\mathtt{F}^n - \mathtt{F}^n\nabla\mathbf{u}^n + \mathbf{u}^{n-1}\cdot\nabla\mathtt{F}^{n-1} + \mathtt{F}^{n-1}\nabla\mathbf{u}^{n-1}, \\ \mathrm{div}\mathbf{u}^n = 0, \end{cases} \tag{4.8}$$

with

$$\begin{aligned} \delta\mathbf{u}^n|_{t=0} = \delta\mathbf{u}^n|_{\partial\Omega} = 0, \\ \delta\mathtt{F}^n|_{t=0} = \delta\mathtt{F}^n|_{\partial\Omega} = 0, \end{aligned}$$

and

$$\int_\Omega \delta P^n dx = 0.$$

Notice that, using Lemma 3.1 and Lemma 3.2,

$$\begin{aligned} &\|-\mathbf{u}^n\cdot\nabla\mathbf{u}^n + \mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^{n-1}\|_{L^p(0,T;L^q(\Omega))} \\ &= \|\delta\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - \mathbf{u}^{n-1}\cdot\nabla\delta\mathbf{u}^{n-1}\|_{L^p(0,T;L^q)} \\ &\le \|\mathbf{u}^{n-1}\|_{L^\infty(0,T;L^q)}\|\nabla\delta\mathbf{u}^{n-1}\|_{L^p(0,T;L^\infty)} + \|\delta\mathbf{u}^{n-1}\|_{L^\infty(0,T;L^q)}\|\nabla\mathbf{u}^n\|_{L^p(0,T;L^\infty)} \\ &\le 4CU^0\left(\|\nabla\delta\mathbf{u}^{n-1}\|_{L^p(0,T;L^\infty)} + T^{\frac{1}{2}-\frac{3}{2q}}\|\delta\mathbf{u}^{n-1}\|_{L^\infty(0,T;L^q)}\right), \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\|-\mathrm{div}(\mathtt{F}^{n\top}\mathtt{F}^n) + \mathrm{div}(\mathtt{F}^{n-1\top}\mathtt{F}^{n-1})\|_{L^p(0,T;L^q(\Omega))} \\ &= \|-\mathrm{div}(\delta\mathtt{F}^{n-1\top}\mathtt{F}^n) - \mathrm{div}(\mathtt{F}^{n-1\top}\delta\mathtt{F}^{n-1})\|_{L^p(0,T;L^q)} \\ &\le \|\mathtt{F}^n\|_{L^\infty(0,T;L^q)}\|\nabla\delta\mathtt{F}^{n-1}\|_{L^p(0,T;L^\infty)} + \|\delta\mathtt{F}^{n-1}\|_{L^\infty(0,T;L^q)}\|\nabla\mathtt{F}^{n-1}\|_{L^p(0,T;L^\infty)} \\ &\quad + \|\mathtt{F}^{n-1}\|_{L^\infty(0,T;L^q)}\|\nabla\delta\mathtt{F}^{n-1}\|_{L^p(0,T;L^\infty)} + \|\delta\mathtt{F}^{n-1}\|_{L^\infty(0,T;L^q)}\|\nabla\mathtt{F}^n\|_{L^p(0,T;L^\infty)} \\ &\le 4CU^0\left(\|\nabla\delta\mathtt{F}^{n-1}\|_{L^p(0,T;L^\infty)} + T^{\frac{1}{2}-\frac{3}{2q}}\|\delta\mathtt{F}^{n-1}\|_{L^\infty(0,T;L^q)}\right), \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\|-\mathbf{u}^n\cdot\nabla\mathtt{F}^n + \mathbf{u}^{n-1}\cdot\nabla\mathtt{F}^{n-1}\|_{L^p(0,T;L^q)} \\ &= \|\mathbf{u}^n\cdot\nabla\delta\mathtt{F}^{n-1} + \delta\mathbf{u}^{n-1}\cdot\nabla\mathtt{F}^{n-1}\|_{L^p(0,T;L^q)} \\ &\le \|\mathbf{u}^n\|_{L^\infty(0,T;L^q)}\|\nabla\delta\mathtt{F}^{n-1}\|_{L^p(0,T;L^\infty)} + \|\delta\mathbf{u}^{n-1}\|_{L^\infty(0,T;L^q)}\|\nabla\mathtt{F}^{n-1}\|_{L^p(0,T;L^\infty)} \\ &\le 4CU^0\left(\|\nabla\delta\mathtt{F}^{n-1}\|_{L^p(0,T;L^\infty)} + T^{\frac{1}{2}-\frac{3}{2q}}\|\delta\mathbf{u}^{n-1}\|_{L^\infty(0,T;L^q)}\right), \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&\| - \mathtt{F}^n\nabla \mathbf{u}^n + \mathtt{F}^{n-1}\nabla \mathbf{u}^{n-1}\|_{L^p(0,T;L^q)} \\
&= \|\mathtt{F}^n\nabla\delta \mathbf{u}^{n-1} + \delta\mathtt{F}^{n-1}\nabla \mathbf{u}^{n-1}\|_{L^p(0,T;L^q)} \\
&\le \|\mathtt{F}^n\|_{L^\infty(0,T;L^q)}\|\nabla\delta\mathbf{u}^{n-1}\|_{L^p(0,T;L^\infty)} + \|\delta\mathtt{F}^{n-1}\|_{L^\infty(0,T;L^q)}\|\nabla\mathbf{u}^{n-1}\|_{L^p(0,T;L^\infty)} \\
&\le 4CU^0\left(\|\nabla\delta\mathbf{u}^{n-1}\|_{L^p(0,T;L^\infty)} + T^{\frac{1}{2}-\frac{3}{2q}}\|\delta\mathtt{F}^{n-1}\|_{L^\infty(0,T;L^q)}\right).
\end{aligned} \tag{4.12}$$

Applying Theorems 3.1-3.2 with the help of (4.9)-(4.12), one deduce that

$$\begin{aligned}
\delta U^n(t) \le 8CU^0\Big(&\|\nabla\delta\mathbf{u}^{n-1}\|_{L^p(0,t;L^\infty)} + \|\nabla\delta\mathtt{F}^{n-1}\|_{L^p(0,t;L^\infty)} \\
&+ t^{\frac{1}{2}-\frac{3}{2q}}(\|\delta\mathtt{F}^{n-1}\|_{L^\infty(0,t;L^q)} + \|\delta\mathbf{u}^{n-1}\|_{L^\infty(0,t;L^q)})\Big).
\end{aligned} \tag{4.13}$$

On the other hand, (4.7) implies that, by Lemma 3.3,

$$\|\delta\mathtt{F}^{n-1}\|_{L^\infty(0,t;L^q)} + \|\delta\mathbf{u}^{n-1}\|_{L^\infty(0,t;L^q)} \le \delta U^{n-1}(t),$$

which, combining with (4.13), Lemma 3.1 and Lemma 3.2 together, gives

$$\delta U^n(t) \le 16CU^0 t^{\frac{1}{2}-\frac{3}{2q}}\delta U^{n-1}(t). \tag{4.14}$$

Thus, if we choose T_0 satisfying (4.5), such that, the condition

$$16CU^0 T_0^{\frac{1}{2}-\frac{3}{2q}} \le \frac{1}{2}$$

is fulfilled, it is clear that $\{(\mathbf{u}^n, \mathtt{F}^n, P^n)\}_{n=1}^\infty$ is a Cauchy sequence in $M^{p,q}_{T_0}$. □

4.4. **The Limit is a solution.** Since $\{(\mathbf{u}^n, \mathtt{F}^n, P^n)\}_{n=1}^\infty$ is a Cauchy sequence in $M^{p,q}_{T_0}$, then it converges. Let $(\mathbf{u}, \mathtt{F}, P) \in M^{p,q}_{T_0}$ be the limit of the sequence $\{(\mathbf{u}^n, \mathtt{F}^n, P^n)\}_{n=1}^\infty$ in $M^{p,q}_{T_0}$. We claim all those nonlinear terms in (4.1) converge to their corresponding terms in (1.5) in $L^p(0,T_0;L^q)$. Indeed, using Lemmas 3.1 and 3.3, we have,

$$\begin{aligned}
&\|\mathbf{u}^n\cdot\nabla\mathbf{u}^n - \mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p(0,T_0;L^q)} \\
&= \|(\mathbf{u}^n - \mathbf{u})\cdot\nabla\mathbf{u}^n + \mathbf{u}\cdot\nabla(\mathbf{u}^n - \mathbf{u})\|_{L^P(0,T_0;L^q)} \\
&\le \|\mathbf{u}^n - \mathbf{u}\|_{L^\infty(0,T_0;L^q)}\|\nabla\mathbf{u}^n\|_{L^p(0,T_0;L^\infty)} + \|\mathbf{u}\|_{L^\infty(0,T_0;L^q)}\|\nabla\mathbf{u}^n - \nabla\mathbf{u}\|_{L^p(0,T_0;L^\infty)} \\
&\le C\|\mathbf{u}^n - \mathbf{u}\|_{M^{p,q}_{T_0}} T_0^{\frac{1}{2}-\frac{3}{2q}} CU^0 + C\|\mathbf{u}\|_{L^\infty(0,T_0;L^q)} T_0^{\frac{1}{2}-\frac{3}{2q}}\|\mathbf{u}^n - \mathbf{u}\|_{M^{p,q}_{T_0}} \\
&\to 0,
\end{aligned}$$

as $n\to\infty$ due to the convergence of $\mathbf{u}^n$ to $\mathbf{u}$ in $M^{p,q}_{T_0}$ and Lemma 3.3. Hence,

$$\mathbf{u}^n\cdot\nabla\mathbf{u}^n \to \mathbf{u}\cdot\nabla\mathbf{u}, \quad \text{in} \quad L^p(0,T_0;L^q).$$

Similarly, we have

$$\begin{aligned}
\operatorname{div}(\mathtt{F}^n\mathtt{F}^{n\top}) &\to \operatorname{div}(\mathtt{F}\mathtt{F}^\top), \quad \text{in} \quad L^p(0,T_0;L^q); \\
\mathbf{u}^n\cdot\nabla\mathtt{F}^n &\to \mathbf{u}\cdot\nabla\mathtt{F}, \quad \text{in} \quad L^p(0,T_0;L^q); \\
\mathtt{F}^n\nabla\mathbf{u}^n &\to \mathtt{F}\nabla\mathbf{u}, \quad \text{in} \quad L^p(0,T_0;L^q).
\end{aligned}$$

Thus, taking the limit as $n\to\infty$ in (4.1), we conclude that (1.5) holds in $L^p(0,T_0;L^q)$, and hence almost everywhere on $\Omega\times[0,T_0]$.

4.5. **Uniqueness.** Let $(\mathbf{u}_1, \mathtt{F}_1, P_1)$ and $(\mathbf{u}_2, \mathtt{F}_2, P_2)$ be two solutions to (1.5) with the initial-boundary conditions (1.6) and (1.7). Denote

$$\delta\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \quad \delta\mathtt{F} = \mathtt{F}_1 - \mathtt{F}_2, \quad \delta P = P_1 - P_2.$$

Note that the triplet $(\delta\mathbf{u}, \delta\mathtt{F}, \delta P)$ satisfies the following system:

$$\begin{cases} \partial_t \delta\mathbf{u} - \mu\Delta\delta\mathbf{u} + \nabla\delta P = -\mathbf{u}_2 \cdot \nabla\delta\mathbf{u} - \delta\mathbf{u} \cdot \nabla\mathbf{u}_1 + \mathrm{div}((\delta\mathtt{F})^\top \mathtt{F}_1 + \mathtt{F}_2^\top \delta\mathtt{F}), \\ \partial_t \delta\mathtt{F} - \Delta\delta\mathtt{F} = -\mathbf{u}_1 \cdot \nabla\delta\mathtt{F} - \delta\mathbf{u} \cdot \nabla\mathtt{F}_2 - \mathtt{F}_1 \nabla\delta\mathbf{u} - \delta\mathtt{F}\nabla\mathbf{u}_2, \\ \mathrm{div}\delta\mathbf{u} = 0, \end{cases} \tag{4.15}$$

with the initial-boundary conditions

$$\delta\mathbf{u}|_{t=0} = \delta\mathbf{u}|_{\partial\Omega} = 0,$$

$$\delta\mathtt{F}|_{t=0} = \delta\mathtt{F}|_{\partial\Omega} = 0,$$

and

$$\int_\Omega \delta P dx = 0.$$

Define

$$\begin{aligned} X(t) :=& \|\delta\mathbf{u}(t)\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} + \|\delta\mathbf{u}\|_{L^p(0,t;W^{2,q})} + \|\partial_t \delta\mathbf{u}\|_{L^p(0,t;L^q)} \\ &+ \|\delta\mathtt{F}(t)\|_{L^\infty(0,t;B_{q,p}^{2(1-\frac{1}{p})})} + \|\delta\mathtt{F}\|_{\mathcal{W}(0,t)}. \end{aligned}$$

Thus, applying Lemmas 3.1 and 3.2 to (4.15), one has, repeating the argument in (4.9)-(4.12),

$$\begin{aligned} X(t) &\le 4CU^0 \Big(\|\nabla\delta\mathbf{u}\|_{L^p(0,t;L^\infty)} + \|\nabla\delta\mathtt{F}\|_{L^p(0,t;L^\infty)} \\ &\quad + t^{\frac{1}{2}-\frac{3}{2q}} (\|\delta\mathtt{F}\|_{L^\infty(0,t;L^q)} + \|\delta\mathbf{u}\|_{L^\infty(0,t;L^q)}) \Big) \\ &\le 16CU^0 t^{\frac{1}{2}-\frac{3}{2q}} X(t) \le \frac{1}{2} X(t). \end{aligned}$$

Hence, $X(t) = 0$ for all $t \in [0, T_0]$, which guarantee the uniqueness on the the interval $[0, T_0]$.

5. Global Existence

In this section, we prove that, if the initial data is sufficiently small, the local solution established in the previous section is indeed global in time. To this end, we first denote by T^* the maximal time of existence for $(\mathbf{u}, \mathtt{F}, P)$. Define the function $H(t)$ as

$$\begin{aligned} H(t) :=& \|\mathbf{u}\|_{L^p(0,t;W^{2,q})} + \|\partial_t \mathbf{u}\|_{L^p(0,t;L^q)} + \|\mathbf{u}\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} \\ &+ \|P\|_{L^p(0,t;W^{2,q})} + \|\mathtt{F}\|_{L^\infty(0,t;B_{q,p}^{2(1-\frac{1}{p})})} + \|\mathtt{F}\|_{\mathcal{W}(0,t)}, \end{aligned}$$

and

$$H_0 := \|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \|\mathtt{F}_0\|_{B_{q,p}^{2(1-\frac{1}{p})} \cap L^q}.$$

To extend the local solution, we need to control the maximal time T^* only in term of the initial data. For this purpose, it is obvious to observe that $H(t)$ is an increasing and continuous function in $[0, T^*)$, and for all $t \in [0, T^*)$, we have, using Lemmas 3.1 and 3.2,

$$H(t) \le C\Big(H_0 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p(0,t;L^q)} + \|\mathrm{div}(\mathtt{F}^\top\mathtt{F})\|_{L^p(0,t;L^q)} + \|\mathbf{u}\cdot\nabla\mathtt{F} + \mathtt{F}\nabla\mathbf{u}\|_{L^p(0,t;L^q)}\Big). \tag{5.1}$$

On the other hand, Lemmas 3.1-3.3 imply that

$$\begin{aligned} \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p(0,t;L^q)} &\le \|\mathbf{u}\|_{L^\infty(0,t;L^q(\Omega))}\|\nabla\mathbf{u}\|_{L^p(0,t;L^\infty)} \\ &\le C\left(\|u_0\|_{L^q} + H(t)\right)H(t)t^{\frac12-\frac{3}{2q}} \\ &\le C(H_0 + H(t))H(t)t^{\frac12-\frac{3}{2q}}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \|\mathrm{div}(\mathtt{F}^\top\mathtt{F})\|_{L^p(0,t;L^q)} &\le C\|F\|_{L^\infty(0,t;L^q)}\|\nabla\mathtt{F}\|_{L^p(0,t;L^\infty)} \\ &\le C(\|\mathtt{F}_0\|_{L^q} + H(t))H(t)t^{\frac12-\frac{3}{2q}} \\ &\le C(H_0 + H(t))H(t)t^{\frac12-\frac{3}{2q}}, \end{aligned} \tag{5.3}$$

and, similarly, by Lemma 3.3,

$$\begin{aligned} &\|\mathbf{u}\cdot\nabla\mathtt{F} + \mathtt{F}\nabla\mathbf{u}\|_{L^p(0,t;L^q)} \\ &\le \|\mathbf{u}\|_{L^\infty(0,t;L^q)}\|\nabla\mathtt{F}\|_{L^p(0,t;L^\infty)} + \|\mathtt{F}\|_{L^\infty(0,t;L^q)}\|\nabla\mathbf{u}\|_{L^p(0,t;L^\infty)} \\ &\le C(\|\mathbf{u}_0\|_{L^q} + H(t))H(t)t^{\frac12-\frac{3}{2q}} + C(\|\mathtt{F}_0\|_{L^q} + H(t))H(t)t^{\frac12-\frac{3}{2q}} \\ &\le C(H_0 + H(t))H(t)t^{\frac12-\frac{3}{2q}}. \end{aligned} \tag{5.4}$$

Substituting (5.2)-(5.4) into (5.1), we get

$$H(t) \le C\left(H_0 + (H_0 + H(t))H(t)t^{\frac12-\frac{3}{2q}}\right). \tag{5.5}$$

Assume that T is the smallest number such that

$$H(T) = 4CH_0.$$

This is possible because $H(t)$ is an increasing and continuous function in time. Then,

$$H(t) < H(T) = 4CH_0, \quad \text{for all} \quad t \in [0, T),$$

and from (5.5), we deduce that

$$3 \le (H_0 + 4CH_0)4CT^{\frac12-\frac{3}{2q}}.$$

Hence, we have

$$T^* > T \ge \left(\frac{3}{8C(H_0 + 4CH_0)}\right)^{\frac{2q}{q-3}}.$$

This implies that the maximal time of existence will go to infinity when the initial data approaches zero. More precisely, we can show that, if the initial data is sufficiently small, the solution exists globally in time. To this end, we need some other estimates for the terms on the right side of (5.1). Indeed, by the imbedding

$$W^{1,q} \hookrightarrow L^\infty,$$

as $q > 3$, we have

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p(0,t;L^q)} &\le \|\mathbf{u}\|_{L^\infty(0,t;L^q(\Omega))}\|\nabla\mathbf{u}\|_{L^p(0,t;L^\infty)}\\ &\le C(\|\mathbf{u}_0\|_{L^q}+H(t))\|\mathbf{u}\|_{L^p(0,t;W^{2,q})}\\ &\le C(H_0+H(t))H(t).\end{aligned}$$

Similarly, we have

$$\|\mathrm{div}(\mathtt{F}^\top\mathtt{F})\|_{L^p(0,t;L^q)} \le C(H_0+H(t))H(t),$$

and

$$\|\mathbf{u}\cdot\nabla\mathtt{F}+\mathtt{F}\nabla\mathbf{u}\|_{L^p(0,t;L^q)} \le C(H_0+H(t))H(t).$$

Thus, (5.1) turns out to be

$$H(t) \le C(H_0+(H_0+H(t))H(t)). \tag{5.6}$$

By the Cauchy-Schwarz inequality, (5.6) becomes

$$H(t) \le C(H_0+H_0^2+2H^2(t)), \tag{5.7}$$

for all $t\in[0,T^*)$. Now we take H_0 sufficiently small such that

$$H_0+H_0^2 \le \delta := \frac{1}{8C^2}. \tag{5.8}$$

Then, under the assumption (5.8), we compute directly from (5.7) and the continuity of $H(t)$ that

$$H(t) \le \frac{1-\sqrt{1-8C^2(H_0+H_0^2)}}{4C} \le \frac{1}{4C}, \tag{5.9}$$

for all $t\in[0,T^*)$. In particular, this implies that

$$\|(\mathbf{u},\mathtt{F},P)\|_{M^{p,q}_{T^*}} \le \frac{1}{4C} < \infty.$$

Hence, according to the local existence in the previous section, we can extend the solution on $[0,T^*)$ to some larger interval $[0,T^*+T_0)$ with $T_0>0$. This is impossible since T^* is already the maximal time of existence. Hence, when the initial data satisfies (5.8), the strong solution is indeed global in time.

The proof of Theorem 2.1 is complete.

6. Weak-Strong Uniqueness

The purpose of this section is to show *Weak-Strong Uniqueness* in Theorem 2.2. To this end, we need to obtain first an energy estimate for the strong solution to the system (1.5). More precisely, we have

Lemma 6.1. *Let p,q satisfy the same conditions as Theorem 2.1 and $(\mathbf{u},\mathtt{F},P)\in M^{p,q}_{T_0}$ be the unique solution to (1.5) on $\Omega\times[0,T_0]$. Then, one has,*

$$\int_\Omega\left(\|\mathbf{u}(t)\|^2+\|\mathtt{F}(t)\|^2\right)dx+\int_0^t\int_\Omega\left(\|\nabla\mathbf{u}\|^2+\|\nabla\mathtt{F}\|^2\right)dxds=\int_\Omega\left(\|\mathbf{u}_0\|^2+\|\mathtt{F}_0\|^2\right)dx.$$

Proof. Note that $\mathbf{u} \in C([0,T_0]; D_{A_q}^{1-\frac{1}{p},p}) \cap L^p(0,T_0; W^{2,q})$ with $q > 3$. Then

$$\mathbf{u} \in C([0,T_0]; L^2) \cap L^2(0,T_0; H^{1+\alpha})$$

for some $\alpha \geq 0$, since

$$D_{A_q}^{1-\frac{1}{p},p} \hookrightarrow B_{q,p}^{2(1-\frac{1}{p})} \cap L^q(\Omega) \hookrightarrow L^2(\Omega),$$

Sobolev's embedding $W^{2,q}(\Omega) \hookrightarrow H^2(\Omega)$ as $q > 3$ and the rest follows from directly the standard interpolation inequality. Similarly,

$$\mathtt{F} \in C([0,T_0]; L^2) \cap L^2(0,T_0; H^{1+\alpha}).$$

Taking the L^2 scalar product in (1.5a) with $\mathbf{u}$ and performing integration by parts, we obtain

$$\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 dx + \int_\Omega |\nabla \mathbf{u}|^2 dx = \int_\Omega \mathtt{F}^\top \mathtt{F} : \nabla \mathbf{u} dx, \tag{6.1}$$

where the notation $A : B$ means the inner product between two matrix, i.e. $A : B = \sum_{i,j} A_{ij}B_{ij}$. Similarly, taking the L^2 inner product in (1.5b) with $\mathtt{F}$ and performing integration by parts, we obtain

$$\frac{d}{dt}\int_\Omega |\mathtt{F}|^2 dx + \int_\Omega |\nabla \mathtt{F}|^2 dx = -\int_\Omega \mathtt{F}\nabla\mathbf{u} : \mathtt{F} dx - \int_\Omega \mathtt{F} : (\mathbf{u}\cdot\nabla\mathtt{F}) dx, \tag{6.2}$$

where $|\mathtt{F}|^2 = \mathtt{F} : \mathtt{F}$ and

$$|\nabla \mathtt{F}|^2 = \sum_{i,j,k} \left|\frac{\partial \mathtt{F}_{ij}}{\partial x_k}\right|^2.$$

Notice that

$$\int_\Omega \mathtt{F} : (\mathbf{u}\cdot\nabla\mathtt{F}) dx = \frac{1}{2}\int_\Omega \mathbf{u}\cdot\nabla|\mathtt{F}|^2 dx = -\frac{1}{2}\int_\Omega \mathrm{div}\mathbf{u}|\mathtt{F}|^2 dx = 0,$$

and, due to $AB : C = A : CB^\top = B : A^\top C$,

$$\int_\Omega \mathtt{F}\nabla\mathbf{u} : \mathtt{F} dx = \int_\Omega \nabla\mathbf{u} : \mathtt{F}^\top\mathtt{F} dx.$$

Hence, adding (6.1) and (6.2) together, we have

$$\frac{d}{dt}\int_\Omega (|\mathbf{u}|^2 + |\mathtt{F}|^2) dx + \int_\Omega (|\nabla\mathbf{u}|^2 + |\nabla\mathtt{F}|^2) dx = 0.$$

Integrating the above equality over time interval $[0,t]$, we obtain the energy equality of this lemma. □

Now, we recall that for the weak solution (v, E, Π) obtained in [16], we have for (almost) all $t \in (0,T)$,

$$\frac{1}{2}\int_\Omega (|v(t)|^2 + |E(t)|^2) dx + \int_0^t \int_\Omega (|\nabla v|^2 + |\nabla E|^2) dx ds \leq \frac{1}{2}\int_\Omega (|\mathbf{u}_0|^2 + |\mathtt{F}_0|^2) dx. \tag{6.3}$$

We remark that, in view of the regularity of $\mathbf{u}$, we deduce from the weak formulation of (1.5) the following equalities:

$$\begin{aligned}&\int_\Omega v\cdot\mathbf{u}dxds+\int_0^t\int_\Omega\nabla\mathbf{u}:\nabla vdxds\\&=\int_\Omega|\mathbf{u}_0|^2+\int_0^t\int_\Omega E^\top E:\nabla\mathbf{u}dxds+\int_0^t\int_\Omega v\cdot\left(\frac{\partial\mathbf{u}}{\partial t}+v\cdot\nabla\mathbf{u}\right)dxds,\end{aligned}\tag{6.4}$$

and

$$\begin{aligned}&\int_\Omega\mathtt{F}:Edx+\int_0^t\int_\Omega\nabla\mathtt{F}:\nabla Edxds\\&=\int_\Omega|\mathtt{F}_0|^2dx-\int_0^t\int_\Omega v\cdot\nabla E:\mathtt{F}dxds-\int_0^t\int_\Omega E\nabla v:\mathtt{F}dxds\\&\quad+\int_0^t\int_\Omega E:\frac{\partial\mathtt{F}}{\partial t}dxds,\end{aligned}\tag{6.5}$$

for a.e. $t\in(0,T)$. Here, we used the identity

$$\int_\Omega v\cdot\nabla\mathbf{u}\cdot wdx=-\int_\Omega v\cdot\nabla w\cdot\mathbf{u}dx,$$

if $\mathrm{div}v=0$.

Since E satisfies the equation (1.5b), we substitute (1.5b) into (6.5), and use the following two facts:

$$\int_0^t\int_\Omega(v\cdot\nabla E:\mathtt{F}+v\cdot\nabla\mathtt{F}:E)dxds=\int_0^t\int_\Omega v\cdot\nabla(E:\mathtt{F})dxds=0,$$

and

$$E\nabla\mathbf{u}:\mathtt{F}+E:\mathtt{F}\nabla\mathbf{u}=\nabla\mathbf{u}:(E^\top\mathtt{F}+\mathtt{F}^\top E),$$

to obtain

$$\begin{aligned}&\int_\Omega\mathtt{F}:Edx+2\int_0^t\int_\Omega\nabla\mathtt{F}:\nabla Edxds\\&=\int_\Omega|\mathtt{F}_0|^2dx-\int_0^t\int_\Omega\nabla\mathbf{u}:(E^\top\mathtt{F}+\mathtt{F}^\top E)dxds\\&\quad+\int_0^t\int_\Omega(v-\mathbf{u})\cdot\nabla\mathtt{F}:Edxds-\int_0^t\int_\Omega\mathtt{F}:E\nabla(v-\mathbf{u})dxds.\end{aligned}\tag{6.6}$$

On the other hand, we can write the equation for $\mathbf{u}$ as

$$\frac{\partial\mathbf{u}}{\partial t}+v\cdot\nabla\mathbf{u}-\Delta\mathbf{u}+\nabla P=(v-\mathbf{u})\cdot\nabla\mathbf{u}-\mathrm{div}(\mathtt{F}^\top\mathtt{F}).\tag{6.7}$$

Multiplying (6.7) by v and integrating over $\Omega \times (0,t)$, we get

$$\begin{aligned}
&\int_0^t\int_\Omega v\cdot\left(\frac{\partial \mathbf{u}}{\partial t}+v\cdot\nabla\mathbf{u}\right)dxds\\
&=-\int_0^t\int_\Omega \nabla\mathbf{u}:\nabla v dxds+\int_0^t\int_\Omega (v-\mathbf{u})\cdot\nabla\mathbf{u}\cdot v dxds\\
&\quad+\int_0^t\int_\Omega \mathtt{F}^\top\mathtt{F}:\nabla v dxds.
\end{aligned}\tag{6.8}$$

Substituting (6.8) into (6.4), we obtain

$$\begin{aligned}
&\int_\Omega \mathbf{u}\cdot v dxds+2\int_0^t\int_\Omega \nabla\mathbf{u}:\nabla v dxds\\
&=\int_\Omega |\mathbf{u}_0|^2+\int_0^t\int_\Omega E^\top E:\nabla\mathbf{u}dxds\\
&\quad+\int_0^t\int_\Omega (v-\mathbf{u})\cdot\nabla\mathbf{u}\cdot v dxds+\int_0^t\int_\Omega \mathtt{F}^\top\mathtt{F}:\nabla v dxds.,
\end{aligned}\tag{6.9}$$

Also, according to Lemma 6.1, we have

$$\frac{1}{2}\int_\Omega(|\mathbf{u}|^2+|\mathtt{F}|^2)dx+\int_0^t\int_\Omega(|\nabla\mathtt{F}|^2+|\nabla\mathbf{u}|^2)dxds=\frac{1}{2}\int_\Omega(|\mathbf{u}_0|^2+|\mathtt{F}_0|^2)dx. \tag{6.10}$$

Summing (6.3), (6.10) and subtracting the sum of (6.6) and (6.9), we obtain for almost all $t\in(0,T)$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega(|\mathbf{u}(t)-v(t)|^2+|\mathtt{F}(t)-E(t)|^2)dx+\int_0^t\int_\Omega(|\nabla\mathbf{u}-\nabla v|^2+|\nabla\mathtt{F}-\nabla E|^2)dxds\\
&\le-\int_0^t\int_\Omega(\mathtt{F}-E)^\top(\mathtt{F}-E):\nabla\mathbf{u}dxds-\int_0^t\int_\Omega(v-\mathbf{u})\cdot\nabla\mathbf{u}\cdot v dxds\\
&\quad-\int_0^t\int_\Omega(v-\mathbf{u})\cdot\nabla\mathtt{F}:E dxds+\int_0^t\int_\Omega \mathtt{F}:E\nabla(v-\mathbf{u})dxds\\
&\quad-\int_0^t\int_\Omega \mathtt{F}^\top\mathtt{F}:\nabla(v-\mathbf{u})dxds\\
&=-\int_0^t\int_\Omega(\mathtt{F}-E)^\top(\mathtt{F}-E):\nabla\mathbf{u}dxds-\int_0^t\int_\Omega(v-\mathbf{u})\cdot\nabla\mathbf{u}\cdot(v-\mathbf{u})dxds\\
&\quad-\int_0^t\int_\Omega(v-\mathbf{u})\cdot\nabla\mathtt{F}:(E-\mathtt{F})dxds+\int_0^t\int_\Omega(E-\mathtt{F})^\top\mathtt{F}:\nabla(v-\mathbf{u})dxds\\
&:=I,
\end{aligned}\tag{6.11}$$

where, we used twice the fact

$$\int_\Omega v\cdot\nabla\mathbf{u}\cdot\mathbf{u}dx=0,$$

if div$v = 0$. For I, we have, by Hölder's inequality,

$$\begin{aligned} |I| \leq & \int_0^t (\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} + \|\nabla \mathtt{F}\|_{L^\infty(\Omega)}) \left(\int_\Omega (|\mathtt{F} - E|^2 + |\mathbf{u} - v|^2) dx \right) ds \\ & + \frac{1}{2} \int_0^t \int_\Omega |\nabla v - \nabla \mathbf{u}|^2 dx ds + C \int_0^t \|\mathtt{F}\|^2_{L^\infty} \|E - \mathtt{F}\|^2_{L^2} ds. \end{aligned} \tag{6.12}$$

Substituting (6.12) back to (6.11), one has

$$\begin{aligned} & \frac{1}{2} \int_\Omega (|\mathbf{u}(t) - v(t)|^2 + |\mathtt{F}(t) - E(t)|^2) dx + \frac{1}{2} \int_0^t \int_\Omega (|\nabla \mathbf{u} - \nabla v|^2 + |\nabla \mathtt{F} - \nabla E|^2) dx ds \\ & \leq \int_0^t (\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} + \|\nabla \mathtt{F}\|_{L^\infty(\Omega)} + C \|\mathtt{F}\|^2_{L^\infty(\Omega)}) \left(\int_\Omega (|\mathtt{F} - E|^2 + |\mathbf{u} - v|^2) dx \right) ds. \end{aligned} \tag{6.13}$$

Notice that

$$\|\nabla \mathbf{u}\|_{L^\infty(\Omega)} + \|\nabla \mathtt{F}\|_{L^\infty(\Omega)} + \|\mathtt{F}\|^2_{L^\infty(\Omega)} \in L^1(0, T).$$

Therefore, using (6.13) together with Grönwall's inequality, we finally conclude that $\mathbf{u} = v$, $\mathtt{F} = E$ $a.e$ and thus $P = \Pi$ in $\Omega \times (0, T)$.

The proof of Theorem 2.2 is complete.

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